Mathematics examination (minor)

Exercise 1 : Study of an operator

Let $C^0(\mathbb{R})$ be the vector space of continuous functions $f : \mathbb{R} \to \mathbb{R}$ and $C^\infty(\mathbb{R})$ be its subspace consisting of smooth functions (that is those having derivatives at all order). We wish to study an operator $L : C^0(\mathbb{R}) \to C^0(\mathbb{R})$ defined as follows : given a function $f\colon \mathbb{R} \to \mathbb{R}$ we set $L(f)\colon \mathbb{R} \to \mathbb{R}$ to be the function given, for any $x \in \mathbb{R}$, by

$$L(f)(x) = f\left(\frac{x+1}{2}\right) + f\left(\frac{x}{2}\right).$$

1. Prove that $C^\infty(\mathbb{R})$ is stable by L, that is if $f \in C^\infty(\mathbb{R})$, then $L(f) \in C^\infty(\mathbb{R})$.
2. Given $\lambda \in \mathbb{R}$, we denote by E_λ the eigenspace of smooth functions with eigenvalue λ, that is the subspace of those functions $f \in C^\infty(\mathbb{R}, \mathbb{R})$ such that

$$\forall x \in \mathbb{R}, \quad L(f)(x) = \lambda\, f(x).$$

   (a) Let f be in E_λ with $|\lambda| > 2$ and let $a \geq 1$. Set

$$M_a = \sup_{x\in[-2a,2a]} |f(x)|.$$

      i. Prove that, for all $x \in [-2a, 2a]$ we have $|L(f)(x)| \leq 2M_a$.

      ii. Prove that $f = 0$.

   (b) Assume now $\lambda \in [-2, 2] \setminus \{0\}$ and f is in E_λ.

      i. Prove that the derivative f' of f is an eigenfunction. Determine the eigenvalue of f'.

      ii. Deduce that f is a polynomial function.

3. We let $(P_n)_{n\in\mathbb{N}}$ be the sequence of polynomial functions given by $P_0(x) = 1$ and

$$P_{n+1}(x) = (n+1)\left(\int_0^x P_n(t)\,dt - \int_0^1 \left(\int_0^u P_n(t)\,dt\right) du\right).$$

   and let also $(H_n)_{n\in\mathbb{N}}$ be the sequence of polynomial functions given by

$$H_n(x) = 2^{1-n}L(P_n)(x) = 2^{1-n}\left(P_n\left(\frac{x+1}{2}\right) + P_n\left(\frac{x}{2}\right)\right).$$

   (a) Prove that $H_0(x) = 1$ and that for $n \geq 1$ and $x \in \mathbb{R}$ one has

$$(i)\ H_n'(x) = nH_{n-1}(x),$$

$$(ii) \int_0^1 H_n(t)\,dt = 0.$$

   (b) Deduce that $H_n = P_n$ for all n.

4. Deduce that for any $i \in \mathbb{N}$, there is a non-zero function f_i such that f_i has eigenvalue 2^{1-i}.

Exercise 2 : quasicommuting matrices

We work over the field $\mathbb{C}$ of complex numbers. Let ω be a complex number. If n is a positive integer and A, B are two matrices in $M_n(\mathbb{C})$, we say that A and B ω-commute if

$$AB = \omega BA.$$

1. (a) Prove that

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

-1-commute.

(b) Assume that A and B ω-commute. Let p and q be two nonnegative integers. Show that A^p and B^q ω^{pq}-commute.

(c) Assume that A and B ω-commute. Let p, q, r and s be nonnegative integers. Find a complex number λ such that $A^p B^q$ and $A^r B^s$ λ-commute.

2. (a) Show that there exist polynomials $Q_{k,N} \in \mathbb{C}[X]$ such that if ω is any complex numbers and A, B ω-commute, the following holds

$$(A + B)^N = \Sigma_{k=0}^{N} Q_{k,N}(\omega) B^k A^{N-k}.$$

What is the value of $Q_{k,N}$ at 0 ? At 1 ?

(b) Let N be a nonnegative integer. We define a polynomial $P_N \in \mathbb{C}[X]$ by the formula

$$P_N(X) = \Pi_{r=1}^{N}(1 + X + \ldots + X^{r-1}).$$

Suppose $k \leq N$. Show that P_k divides P_N.

(c) Using induction on N, show that

$$P_k P_{N-k} Q_{k,N} = P_N.$$

(d) Assume that ω is a primitive N-th root of unity, that is, $\omega^N = 1$ and $\omega^k \neq 1$ for all $k < N$. Show that

$$(A + B)^N = A^N + B^N.$$